E. B. Frost, Director of the Yerkes Observatory, Williams Bay, Wisconsin, U.S.A.;

The Rev. J. G. Hagen, S.J., Director of the Vatican Observatory, Rome; and

Johannes Franz Hartmann, Astrophysical Observatory, Potsdam, Germany.

Seventy-one presents were announced as having been received since the last meeting, including, amongst others:—

J. G. Böhn, die Kunst-Uhren auf der K.-K. Sternwarte zu Prag, presented by the Prague Observatory; The History of the Geological Society of London, by H. B. Woodward, presented by the Society.

Sixteen charts of the Astrographic Chart of the heavens, presented by the Royal Observatory, Greenwich.

On the Lunar Inequalities due to the Motion of the Ecliptic and the Figure of the Earth. By Ernest W. Brown, Sc.D., F.R.S.

§ 1. *The general disturbing function for the motion of the ecliptic.*—Let θ_1, θ_2, θ_3 be the angular velocities of a set of moving rectangular axes about themselves; x, y, z the co-ordinates; u, v, w the velocities of a particle with respect to these axes; and let F be the force function divided by the mass of the particle. Then the equations of motion are given by

$$\frac{du}{dt} - v\theta_3 + w\theta_2 = \frac{\partial F}{\partial x},$$

$$\frac{dv}{dt} - w\theta_1 + u\theta_3 = \frac{\partial F}{\partial y},$$

$$\frac{dw}{dt} - u\theta_2 + v\theta_1 = \frac{\partial F}{\partial z},$$

where

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

Put

$$H = \tfrac{1}{2}(u^2 + v^2 + w^2) - F - R,$$

where

$$R = vx\theta_3 - wx\theta_2 + wy\theta_1 - uy\theta_3 + uz\theta_2 - vz\theta_1;$$

then, if we assume that θ_1, θ_2, θ_3 are independent of x, y, z, u, v, w, we may write the equations of motion in the canonical form,

$$\frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dx}{dt} = \frac{\partial H}{\partial u},$$

$$\frac{dv}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v},$$

$$\frac{dw}{dt} = -\frac{\partial H}{\partial z}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}.$$

If R be neglected, the equations become of the same form as for fixed axes; hence R is the disturbing function for the motions of the axes. These motions are so small that we can neglect squares and higher products of the terms in R, and therefore can treat R as a disturbing function of the same nature as that used for ordinary planetary perturbations.

In order to obtain θ_1, θ_2, θ_3, we put i' for the inclination of the moving ecliptic (xy plane) to the fixed ecliptic (that of 1850·0), τ for the longitude of the node on the fixed ecliptic, L for the angle which the zx plane makes with the plane containing the poles of the fixed and moving ecliptics. Then, by Euler's equations,

$$\theta_1 = \frac{di'}{dt} \sin L - \sin i' \cos L \frac{d\tau}{dt},$$

$$\theta_2 = \frac{di'}{dt} \cos L + \sin i' \sin L \frac{d\tau}{dt},$$

$$\theta_3 = \frac{d\tau}{dt} \cos i' + \frac{dL}{dt}.$$

As L is at our disposal, we so take it that the distances of the origins of reckoning on the fixed and moving ecliptics from their common node are the same, that is, so that $L = 90° - \tau$. The moving axis of x therefore passes through a "departure point."

When we are given the values of i', τ in terms of the time, we have the material necessary for the solution of the problem.

2. As a matter of fact, i', $\frac{di'}{dt}$, $\frac{d\tau}{dt}$ are so small that their squares and products may be neglected. We can therefore put

$$\sin i' = i', \ \cos i' = 1, \ \tau = \text{const.},$$

and obtain

$$\theta_1 = \frac{di'}{dt} \cos \tau, \quad \theta_2 = \frac{di'}{dt} \sin \tau, \quad \theta_3 = 0.$$

Whence

$$R = \frac{di'}{dt}\left\{ (wy - vz)\cos\tau + (uz - wx)\sin\tau \right\},$$

or, since u differs from $\frac{dx}{dt}$ by a quantity of the same order as i', etc.,

$$R = \frac{di'}{dt}\left\{ \left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\cos\tau - \left(x\frac{dz}{dt} - z\frac{dx}{dt}\right)\sin\tau \right\}$$

$$= \frac{di'}{dt}Q, \text{ suppose.}$$

The method of the variation of arbitrary constants permits us to substitute in Q the values of x, y, z, and of their derivatives in terms of the arbitrary constants and the time as found in the solution of the main problem of the lunar theory. These values can be obtained directly from the results which I have given in my papers in the *Memoirs* of the Society on this subject.

3. The value of i' is of the form

$$i' = pt + P,$$

where p is a constant and P a sum of periodic terms whose arguments depend on those of the motions of the Earth and planets. The latter terms have generally been neglected, but I shall show that they give rise to a few terms which are not insensible. At this stage the question arises as to what moving ecliptic we shall refer the motion of the Moon. We may refer it to the *actual* ecliptic, in which case I have found by calculation that P gives rise to a number of primary terms of short period and a few of long period. Or we may refer it to the *mean* ecliptic, *i.e.* put $P=0$ in the above formulæ; but if we do so, it will be necessary to introduce the latitude of the Sun above this mean ecliptic into F. With this latter method, I have found that all the primary short-period terms practically disappear and the long-period terms have rather smaller coefficients than with the former plane of reference.

I therefore adopt the mean ecliptic together with the one or two very minute terms which are of long period relative to that of the Moon's node, and which do not then give rise to any terms in the Moon's co-ordinates. Hence, for the computation, we may put

$$R_1 = pQ.$$

4. The principal part of the force function for the Moon's motion under the influence of the Earth and Sun is

$$F = \frac{\mu}{r} + \frac{m'}{r'^3}\left\{ \frac{3}{2}\frac{(xx' + yy' + zz')^2}{r'^2} - \frac{1}{2}(x^2 + y^2 + z^2) \right\}$$

where x', y', z' are the co-ordinates of the Sun.

The main problem is solved with $z'=0$. Hence the additional portion due to z' is, if we neglect squares of z',

$$R_2 = \frac{m'}{r'^3} \cdot \frac{3(xx'+yy')zz'}{r'^2}.$$

With the notation and limitations of § 2, 3 we have

$$z' = i'(y' \cos \tau - x' \sin \tau).$$

Also, with this value for z' and by means of the equations $\frac{d^2x}{dt^2} = \frac{\partial F}{\partial x}$, etc., it is easy to show that

$$\frac{m'}{r'^3} \frac{3(xx'+yy')zz'}{r'^2} = i' \left\{ \left(z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2} \right) \cos \tau - \left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2} \right) \sin \tau \right\}.$$

Hence, since τ is a constant, we obtain

$$R_2 = -i'\frac{dQ}{dt},$$

where for i' we substitute its periodic portions, omitting those mentioned at the end of § 3.

5. It is of some interest to compare the two disturbing functions R_1, R_2. In general we have

$$R_1 = Q\frac{di'}{dt}, \quad R_2 = -i'\frac{dQ}{dt} = Q\frac{di'}{dt} - \frac{d}{dt}(i'Q).$$

Thus the effect of the rotation of the axes alone, apart from the motion of the Sun out of the fixed plane, is to introduce a term $\frac{d}{dt}(i'Q)$. From these expressions we may also prove the statements made in § 3 with reference to the presence of long- and short-period terms, remembering the two forms which have been given for $\frac{dQ}{dt}$. In fact, the terms of long period in Q are relatively much larger than those in $\frac{dQ}{dt}$ owing to the presence of small divisors; for a term of *very* long period in R_2, the portion arising from $\frac{d}{dt}(i'Q)$ is small compared with that arising from R_1.

6. The computation from the two functions R_1, R_2, as defined in §§ 3, 4, give the results which follow. The values of Leverrier * have been used for i', τ, and the method is the same as that which I have used for planetary inequalities in general.

I put w_1, w_2, w_3 for the mean longitudes of the Moon, of its perigee and of its node; l, D, F, n, e, γ are the quantities as defined by Delaunay; T, V, J are the mean longitudes of the

* *Ann. Obs. Paris (Mém.)*, vol. iv. pp. 13–21.

Earth, Venus, and Jupiter, respectively; $\phi = w_3 + 96^\circ\!\cdot\!2$; $\phi' = w_1 + 96^\circ\!\cdot\!2$.

Then the variations of the elements for the principal inequality, argument ϕ, arising from R_1, are

$$\begin{aligned} \delta w_1 &= -\ 0''\!\cdot\!289 \sin\phi, & \delta n &= +0''\!\cdot\!0014 n \cos\phi, \\ \delta w_2 &= +\ 0''\!\cdot\!840 \sin\phi, & \delta e &= \quad \cdot 000 \ \cos\phi, \\ \delta w_3 &= -15''\!\cdot\!59 \ \sin\phi, & \delta\gamma &= +0''\!\cdot\!698 \ \cos\phi. \end{aligned}$$

These agree very closely with the values found by Professor Newcomb.* They differ a little more from those found by Dr. Hill; † he, however, used a method with a literal development, and the small differences are probably due to slow convergence.

The principal term in latitude, argument ϕ', has a coefficient of $+1''\!\cdot\!382$.

7. The new terms arising from R_2 are, if we omit terms with coefficients less than $0''\!\cdot\!010$,

In Longitude.	In Latitude.
$+0''\!\cdot\!019 \sin(\phi + 5T - 3V + 119^\circ)$	$+0''\!\cdot\!077 \sin(\phi' + 5T - 3V + 119^\circ\!\cdot\!4)$
$+0''\!\cdot\!003 \sin(\phi + 2w_2 - 2J + 90^\circ)$,	$+0''\!\cdot\!030 \sin(\phi' - 5T + 3V - 119^\circ\!\cdot\!4)$
	$+0''\!\cdot\!035 \sin(\phi' + 2J + 72^\circ)$
	$+0''\!\cdot\!018 \sin(\phi' - 2J - 72^\circ)$.

The two terms in longitude are "primary" terms of long period. The second one is of interest, since its period is that of the empirical term which appears to represent best the long-period difference between theory and observation, namely, about 280 years. Its coefficient is, however, too small to account for this difference.‡ There is a term with the same argument due to the direct effect of the planets, but its coefficient is also insensible. The terms in latitude are "secondary," and have periods approximating to a month.

8. *Inequalities arising from the figure of the Earth.*—These have been computed very fully by Dr. Hill,§ who used the method of Delaunay, and there would be no reason for the mention of them here if it were not that slight doubts have arisen concerning their degrees of accuracy on account of slow convergence. I have used the equations of variations and the results of my lunar theory which allow the coefficients to be obtained easily within $0''\!\cdot\!003$. This method is brief, and the computations occupied about eight

* Carnegie Institute, publ. 72, p. 132. The difference in the value of δn is due to a difference in the definition of this variation.

† *Annals of Math.*, vol. i. p. 57. Coll. Works, vol. ii. p. 77.

‡ In a note on p. 170 of the present volume, and in this paper as read before the Society, I gave $0''\!\cdot\!21$ as the value of this coefficient. A recalculation with the first form of R_2 revealed an error in the former computation.

§ *Amer. Eph. Papers*, vol. iii. pp. 201–344. Coll. Works, vol. ii. pp. 181–320.

days. I put $\psi = w_3 +$ the precession, and obtain for the variations of the elements

$$\begin{aligned} \delta w_1 &= +\ 7''\cdot 317 \sin\psi, & \delta n &= -0''\cdot 009 \cos\psi, \\ \delta w_2 &= -\ 2''\cdot 092 \sin\psi, & \delta e &= +0''\cdot 002 \cos\psi, \\ \delta w_3 &= +96''\cdot 69\ \sin\psi, & \delta\gamma &= -4''\cdot 351 \cos\psi. \end{aligned}$$

The short-period terms add $-0''\cdot 017 \sin(\psi + F)$ to the latitude. The principal term in longitude is δw_1, and the principal elliptic terms are $+0''\cdot 519 \sin(\psi + l) + 0''\cdot 515 \sin(\psi - l)$. The principal term in latitude is $-8''\cdot 355 \sin(\psi + F)$, and that with argument $\psi - F$ is $+0''\cdot 338 \sin(\psi - F)$. When Hill's results are reduced to the value of the ellipticity used above (1/296·3),* my coefficient of the principal term in latitude agrees with his, and my coefficient of the principal term in longitude is $0''\cdot 030$ less. The coefficients for the term with argument $\psi - F$ also agree; this fact furnishes a useful test, since this coefficient is the difference of two numbers each nearly thirteen times as large as the coefficient. The two principal elliptic terms are $0''\cdot 021$, $0''\cdot 017$, respectively, greater than those of Hill. An examination of Hill's literal developments for the coefficients shows that the small differences can all be explained by slow convergence.

The complete results for the classes of terms considered here will be given in the fifth (concluding) part of my *Theory of the Motion of the Moon*, with the terms due to the action of the planets and to perturbations not considered in the previous parts.

New Haven, Conn.:
1908 *February* 1.

Postscript (1908 *April* 4).—The terms of the second order have been examined and produce nothing sensible. However, in consequence of my attention being called by Dr. Hill to a doubt as to whether there was a portion of the secular acceleration due to the figure of the Earth terms being referred to a moving ecliptic, I made an actual computation of this second order perturbation and found that the greatest effect could be exhibited in the form of a term of period about 15,000 years and coefficient $0''\cdot 15$. This, equivalent to a secular acceleration of $0''\cdot 0001$ within historic times, is entirely insensible. The computation will be given in chapter xiv. of my memoir just mentioned.

* Corresponding to the result marked (β) in *M.N.*, vol. lxiv. p. 531.